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CATEGORICAL SHAPE THEORY AND THE BACK AND FORTH PROPERTY

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Categorical shape theory begins with the simple observation that corresponding to each functor $K:\mathcal{P}\to\mathcal{T}$ there is a coshape category $\mathcal{S}_K$ and a shape category $\hat{\mathcal{S}}_K$. Both categories have the same objects as $\mathcal{T}$ and the hom set $\mathcal{S}_K(X,Y)$ is simply the collection of natural transformations $\mathcal{T}(K-,X)\to\mathcal{T}(K-,Y)$ with $\hat{\mathcal{S}}_K(X,Y)$ the transformations $\mathcal{T}(Y,K-)\to\mathcal{T}(X,K-)$. There are canonical functors $S:\mathcal{T}\to\mathcal{S}_K$ and $\hat{S}:\mathcal{T}\to\hat{\mathcal{S}}_K$.

If $K:\mathcal{P}\to\mathcal{T}$ has an adjoint, then the shape category is isomorphic to the Kleisli category. In many other cases it may be readily identified also. For example, suppose that $\mathcal{T}$ is the homotopy category and $K:\mathcal{P}\to\mathcal{T}$ is the inclusion of a one object full subcategory $\mathcal{P}$. Then $S:\mathcal{T}\to\mathcal{S}_K$ is essentially the homotopy functor $\pi_n(-)$ if $\mathcal{P}=\{S^n\}$ and $\hat{S}:\mathcal{T}\to\hat{\mathcal{S}}_K$ the cohomology functor $H^n(-;\pi)$ if $\mathcal{P}=\{K(\pi,n)\}$.

Deleanu–Hilton [6] and LeVan [8] have successfully shown that work initiated by Borsuk [5] on topological shape and continued by Mardesic–Segal [11] fits into this general setting for $K:\mathcal{P}\to\mathcal{T}$ the restriction to compact Hausdorff spaces of the inclusion of the full subcategory of spaces of the homotopy type of a CW complex in the homotopy category. A list of the more than 300 references to the topological theory may be found in Segal [12].

The purpose of this paper is to show that certain classical results from algebra and logic fit into the same abstract setting and thus make the general theory available to those disciplines. In this way we can make precise the fact that the relationship between the Warsaw circle (of Borsuk) and S^1 is of the same type as that holding between two reduced primary torsion abelian groups with the same Ulm invariants or between two superatomic Boolean algebras having the same Day invariants. In fact shape theoretic arguments in algebra really date back to Cantor who essentially had a proof that dense linearly ordered sets without endpoints are isomorphic in a suitable $\mathcal{S}_K$ (cf. [3]).

We recall from MacLane [9] that the objects of the comma category $(K\downarrow X)$ are the $\mathscr{T}$ morphisms $KP \to X$ and that there is a canonical functor $D_X : (K \downarrow X) \to \mathscr{P}$ with $D_X(KP \to X) = P$.

It can readily be shown that the coshape category $\mathscr{S}_K$ is isomorphic to one in which functors $\theta : (K \downarrow X) \to (K \downarrow Y)$ with $D_Y\theta = D_X$ correspond to $\mathscr{S}_K$ morphisms $X \to Y$. It is thus not too surprising that in their natural setting it can be a problem to recognize shape morphisms as such.

In fact this *problem of recognition* is nontrivial since it amounts to showing that certain functors are initial or final (in the sense of [9]) and this is precisely the place where work special to the category is involved.

For example, in the topological applications of [6] and [8] it is crucial for shape isomorphism to verify initiality of $U_X : I_X \to (X \downarrow K)$ where (a) I_X = the codirected set of minimal covers C of X and U_X takes C to the canonical map $X \to$ nerve C *or*

(b) I_X is as in (a) and U_X takes C to the canonical map $X \to$ Vietoris complex *or*

(c) I_X = the neighborhood filter base and U_X associates with each neighborhood the embedding of X in that neighborhood.

As an example in the algebraic case we let $K^* : \mathscr{P}^* \to \mathscr{T}^*$ be the inclusion functor where $\mathscr{T}^*$ is the category of all reduced primary torsion abelian groups and $\mathscr{P}^*$ is the full subcategory of finite such groups. For shape isomorphism as described in Section 3 it is crucial to verify finality for $U_X : \Phi \to (K \downarrow X)$ where Φ is a certain category whose objects are height preserving automorphisms between finite subgroups of X and Y and $U_X(\phi)$ is the class of the inclusion of K (domain ϕ) in X and $K : \mathscr{P} \to \mathscr{T}$ is the induced functor between preorders associated with the subcategories of monomorphisms of $\mathscr{P}^*$ and $\mathscr{T}^*$.

In this paper we show explicitly how the back and forth property for a family $|\Phi|$ of isomorphisms, as described by Barwise [3] and Barwise–Ekloff [4] corresponds to the existence of a shape isomorphism in a certain category $\mathscr{S}_K$.

The first step in recognizing a shape map is to show how a morphism in the Pro-$\mathscr{P}$ category is induced. Accordingly the first section shows how to get a Pro-$\mathscr{P}$ isomorphism out of a family of isomorphisms in a certain category $\mathscr{P}^*$. The second section shows how to get suitable final functors if $|\Phi|$ has domain and range enlargement properties. The last section shows how to put together the cofinal functors and a Pro-$\mathscr{P}$ isomorphism to get a shape isomorphism. The algebraic example described above is treated in extenso throughout in order to illustrate the development of the theory. Other examples may be found in the last section.

This paper also serves to give a precise meaning to the statement of Barwise in [3] that "any attempt to define an absolute notion of isomorphism stronger than $\cong_p$ must fail". By $X \cong_p Y$ he means that there exists a family $|\Phi|$ of isomorphisms between finite subobjects of X and Y having the back and forth property. In terms of shape theory his statement may be regarded as a reflection of the fact that $S \circ K : \mathscr{P} \to \mathscr{T} \to \mathscr{S}_K$ is dense (in the sense of MacLane [9]) for K full or simply that the coshape category of K equals that of $S \circ K$. Furthermore it shows that indeed "$\cong_p$ is a very natural notion of isomorphism, one of which mathematicians should be aware" because it is a notion of coshape isomorphism for a certain functor K

associated with the inclusion of the full subcategory of finitely generated algebraic systems $\langle A, R, F\rangle$ of type τ (in the sense of Mal'cev [10]) in the category of all such systems.

1. The Pro-$\mathscr{P}$ morphism induced by a family of isomorphisms

In this section we show how a family $|\Phi|$ of isomorphisms between subobjects of given objects X and Y in a category $\mathscr{T}^*$ induces an isomorphism in a certain category Pro-$\mathscr{P}$. For example, let $\mathscr{T}^*$ be the category of reduced torsion abelian p-groups for a fixed prime p. If $|\Phi|$ is any family of isomorphisms whose domains and ranges are finite subgroups of X and Y, respectively, then a Pro-$\mathscr{P}$ isomorphism is induced between certain domain and range functors associated with $|\Phi|$. In this example, $\mathscr{P}$ is the preorder obtained from the subcategory $\tilde{\mathscr{P}}$ of monomorphisms in the full subcategory $\mathscr{P}^*$ of finite groups belonging to $\mathscr{T}^*$ and $K^* : \mathscr{P}^* \to \mathscr{T}^*$ is the inclusion functor. In the following sections it will be shown that if there is a family $|\Phi|$ satisfying in addition certain cofinality properties, as is the case when X and Y have the same Ulm invariants, then the Pro-$\mathscr{P}$ isomorphism obtained induces a shape isomorphism between X and Y.

Let

$$\begin{tikzcd} \mathscr{P}^* \arrow[rr, "K^*"] \arrow[dr, "U"'] & & \mathscr{T}^* \arrow[dl, "U"] \\ & \text{Sets} & \end{tikzcd} \tag{1.1}$$

be a commutative diagram of categories and functors such that U is faithful and preserves and reflects monomorphisms. Let $\tilde{K} : \tilde{\mathscr{P}} \to \tilde{\mathscr{T}}$ be the restriction of K^* to the subcategories $\tilde{\mathscr{P}}$ and $\tilde{\mathscr{T}}$ of monomorphisms in $\mathscr{P}^*$ and $\mathscr{T}^*$ respectively. A morphism γ of $\mathscr{P}^*$ or $\mathscr{T}^*$ is called an *inclusion* if $U\gamma$ is an inclusion in Sets. We assume that every morphism of $\tilde{\mathscr{T}}$ can be factored uniquely as an isomorphism followed by an inclusion where for $\gamma : \tilde{K}P \to X$ this factorization takes the special form $\gamma = \gamma' \circ \tilde{K}\rho : \tilde{K}P \to \tilde{K}P' \to X$ for ρ a $\mathscr{P}^*$ isomorphism.

Suppose that $\tilde{\mathscr{P}}^2$ is the category whose objects are $\tilde{\mathscr{P}}$ morphisms $\phi_i : D\phi_i \to R\phi_i$ and whose morphisms $\phi_i \to \phi_j$ are those pairs (α, β) of $\tilde{\mathscr{P}}$ morphisms for which

$$\begin{tikzcd} D\phi_i \arrow[r, "\phi_i"] \arrow[d, "\alpha"'] & R\phi_i \arrow[d, "\beta"] \\ D\phi_j \arrow[r, "\phi_j"'] & R\phi_j \end{tikzcd} \tag{1.2}$$

commutes in $\tilde{\mathscr{P}}$.

Let Φ be a full subcategory of $\tilde{\mathscr{P}}^2$ whose objects $\phi_i : D\phi_i \to R\phi_i$ are $\tilde{\mathscr{P}}$-isomorphisms whose domains and ranges are all such that there are inclusions $\iota_X : \tilde{K}D\phi_i \to X$ and $\iota_Y : \tilde{K}R\phi_i \to Y$ for fixed objects X and Y of $\tilde{\mathscr{T}}$. It is convenient to work with the following lattice of subcategories of Φ

$$
\begin{tikzcd}
 & \Phi & \\
\Phi_D \arrow[ur] & & \Phi_R \arrow[ul] \\
\Phi'_D \arrow[u, "I_D"] & & \Phi'_R \arrow[u, "I_R"'] \\
 & \Phi_I \arrow[ul] \arrow[ur] &
\end{tikzcd}
\tag{1.3}
$$

all having the same objects. A Φ morphism $(\alpha, \beta): \phi_i \to \phi_j$ is in Φ_D if $U\alpha$ is an inclusion and in Φ'_D if in addition there is an inclusion $U\gamma : UR\phi_i \to UR\phi_j$ where we note that the latter inclusion may differ from $U\beta$. Similarly (α, β) is in Φ_R if $U\beta$ is an inclusion and in Φ'_R if in addition there is an inclusion $U\delta : UD\phi_i \subseteq UD\phi_j$. Finally, Φ_I is just $\Phi_D \cap \Phi_R$. The commutativity required by (1.2) shows that all the categories of (1.3) are preorders, except possibly for Φ.

In other words a given family $|\Phi|$ of isomorphisms between subobjects of X and Y may be considered as the set of objects of each of the categories appearing in (1.3). We next define the Pro category and show that there is a Pro-$\mathscr{P}$ isomorphism connecting the domain functor of Φ'_D and the range functor of Φ'_R.

Let $\mathscr{P}$ be the quotient preorder of $\bar{\mathscr{P}}$ with quotient functor $Q : \bar{\mathscr{P}} \to \mathscr{P}$, that is, $\mathscr{P}$ and $\bar{\mathscr{P}}$ have the same objects and the hom set $\mathscr{P}(A, B)$ has one element if there is a $\bar{\mathscr{P}}$ morphism $A \to B$ and is otherwise empty. The functor $\bar{K} : \bar{\mathscr{P}} \to \bar{\mathscr{T}}$ induces a functor $K : \mathscr{P} \to \mathscr{T}$ between quotient preorders, called the functor *associated* with K^*. We let $\bar{D} : \Phi \to \bar{\mathscr{P}}$ and $D = Q\bar{D}$ be the domain functors and $\bar{R}$ and $R = Q\bar{R}$ be the range functors. Furthermore let $D' = DI_D$ and $R' = RI_R$.

For any category $\mathscr{C}$ we let Pro-$\mathscr{C}$ be the category whose objects are functors with range $\mathscr{C}$. If $F : \mathscr{I} \to \mathscr{C}$ and $G : \mathscr{J} \to \mathscr{C}$ are such objects, then a morphism $f : F \to G$ in Pro-$\mathscr{C}$ is an equivalence class $f = \{\psi, (f_i)_{i \in |\mathscr{I}|}\}$ where $\psi : |\mathscr{I}| \to |\mathscr{J}|$ is a function and for each $i \in |\mathscr{I}|$, $f_i : Fi \to G\psi i$. Furthermore it is required that for each family $r_\lambda : i_\lambda \to i$ in $\mathscr{I}$ indexed by $\lambda = 1, 2, \ldots, n$, there exists $j \in |\mathscr{J}|$ and morphisms $t : \psi i \to j$ and $s_\lambda : \psi i_\lambda \to j$, $\lambda = 1, \ldots, n$, such that

$$
\begin{tikzcd}
Fi_\lambda \arrow[r, "f_{i_\lambda}"] \arrow[dd, "Fr_\lambda"'] & G\psi i_\lambda \arrow[dr, "Gs_\lambda"] & \\
 & & Gj \\
Fi \arrow[r, "f_i"'] & G\psi i \arrow[ur, "Gt"'] &
\end{tikzcd}
\tag{1.4}
$$

commutes in $\mathscr{C}$ for each $\lambda = 1, 2, \ldots, n$. Furthermore $(\psi, (f_i)) \sim (\eta, (g_i))$ if for each $i \in |\mathscr{I}|$ there exists an object k in $\mathscr{J}$ and morphisms $s : \psi i \to k$, $u : \eta i \to k$ such that

$$
\begin{tikzcd}
 & G\psi i \arrow[dr, "Gs"] & \\
Fi \arrow[ur, "f_i"] \arrow[dr, "g_i"'] & & Gk \\
 & G\eta i \arrow[ur, "Gu"'] &
\end{tikzcd}
\tag{1.5}
$$

commutes in $\mathscr{C}$.

The category Pro-$\mathscr{C}$ just described is dual to the one given by Deleanu–Hilton [6] which itself extended the description of Artin–Mazur [2].

Theorem 1.6. *There is a Pro-$\mathscr{P}$ isomorphism g from $D' : \Phi'_D \to \mathscr{P}$ to $R' : \Phi'_R \to \mathscr{P}$ corresponding to each family $|\Phi|$ of $\tilde{\mathscr{P}}$ isomorphisms between subobjects of X and Y in $\tilde{\mathscr{F}}$.*

Proof. The Pro-$\mathscr{P}$ isomorphism $g = \{\psi, (g_{\phi_i})_{\phi_i \in |\Phi|}\}$ is defined in the following way. Let $\psi : |\Phi'_D| \to |\Phi'_R|$ be the identity function. For each $\phi_i : D\phi_i \to R\phi_i$ in $|\Phi'_D|$ let $g_{\phi_i} = [\phi_i] : D\phi_i \to R\psi\phi_i = R\phi_i$ be the unique $\mathscr{P}$ morphism determined by the $\tilde{\mathscr{P}}$ isomorphism ϕ_i. Furthermore given a morphism $(\alpha, \beta) : \phi_i \to \phi_j$ in Φ'_D, with corresponding diagram

$$\begin{tikzcd} D\phi_i \arrow[r, "\phi_i"] \arrow[d, "\alpha = \subseteq"'] & R\phi_i \arrow[d, "\beta"] \\ D\phi_j \arrow[r, "\phi_j"'] & R\phi_j \end{tikzcd} \tag{1.7}$$

commuting in $\tilde{\mathscr{P}}$ there exists $j = \psi\phi_j = \phi_j$ in $|\Phi'_R|$ and Φ'_R morphisms $t = 1_j : \psi\phi_j \to j$ and $s_\lambda = s = (\alpha', \beta') : \phi_i \to \phi_j$ with corresponding diagram

$$\begin{tikzcd} D\phi_i \arrow[r, "\phi_i"] \arrow[d, "\alpha' = \phi_j^{-1}\beta'\phi_i"'] & R\phi_i \arrow[d, "\beta' = \subseteq"] \\ D\phi_j \arrow[r, "\phi_j"'] & R\phi_j \end{tikzcd} \tag{1.8}$$

commuting in $\tilde{\mathscr{P}}$. Such a β' exists because U reflects monomorphisms and $(\alpha, \beta) \in \Phi'_D$ implies that there is an inclusion $U\beta' : UR\phi_i \subseteq UR\phi_j$. Furthermore, U faithful implies that β' is unique. In this setting diagram (1.4) becomes

$$\begin{tikzcd} D\phi_i \arrow[r, "{[\phi_i]}"] \arrow[d, "{D'(\alpha,\beta) = [\alpha]}"'] & R\phi_i \arrow[d, "{R'(\alpha',\beta') = [\beta']}"] \\ D\phi_j \arrow[r, "{[\phi_j]}"'] & R\phi_j \end{tikzcd} \tag{1.9}$$

which is in $\mathscr{P}$ and thus commutes. The inverse $g^{-1} : R' \to D'$ of g is determined in the same way by $|\Phi|^{-1} = \{\phi_i^{-1} \mid \phi_i \in |\Phi|\}$.

Remark 1.10. A Pro-$\tilde{\mathscr{P}}$ isomorphism is obtained using the same proof if for any two objects f and g of Φ either $f \subseteq g$ or ($f \not\subseteq g$ and $Df \not\subseteq Dg$ and $Rf \not\subseteq Rg$). In general there does not exist a Pro-$\tilde{\mathscr{P}}$ isomorphism $\tilde{D}' = \tilde{D}I_D \to \tilde{R}I_R = \tilde{R}'$ by (3.7) and (3.15). If we dropped the requirement that Φ be full in $\tilde{\mathscr{P}}^2$, then we could always get a Pro-$\tilde{\mathscr{P}}$ isomorphism by allowing only inclusion morphisms in Φ. However, the fullness is needed in the next section.

2. Finality and enlargement

The purpose of this section and the next one is to show how a family $|\Phi|$ of isomorphisms between subobjects of X and Y may yield a shape isomorphism. The method used in the next section requires that we verify that certain functors associated with $|\Phi|$ are final. This section defines those functors and shows that they are final provided that certain enlargement properties are valid in the category Φ associated with $|\Phi|$.

Let the functor $T_X : \Phi_D \to (\tilde{K} \downarrow X)$ be defined on objects by $T_X(\phi_i : D\phi_i \to R\phi_i) = (\iota_X : \tilde{K}D\phi_i \to X)$ and on morphisms by $TX(\alpha, \beta) = \alpha$. It is clear that the $\tilde{\mathcal{P}}$ morphism $\alpha : D\phi_i \to D\phi_j$ can be regarded as a $(\tilde{K} \downarrow X)$ morphism since

$$\begin{tikzcd} KD\phi_i \arrow[rr, "\tilde{K}\alpha"] \arrow[dr, "\iota_X"'] & & \tilde{K}D\phi_j \arrow[dl, "\nu_X"] \\ & X & \end{tikzcd} \tag{2.1}$$

is a diagram of inclusions in $\tilde{\mathcal{T}}$ which must commute since U is faithful. Clearly the diagram

$$\begin{tikzcd} \Phi_D \arrow[rr, "T_X"] \arrow[dr, "\tilde{D}"'] & & (\tilde{K} \downarrow X) \arrow[dl, "\tilde{D}_X"] \\ & \tilde{\mathcal{P}} & \end{tikzcd} \tag{2.2}$$

commutes where $\tilde{D}_X$ is the projection functor defined by $\tilde{D}_X(\gamma : \tilde{K}A \to X) = A$ and $\tilde{D}_X(\alpha : \gamma \to \delta) = \alpha : A \to B$ in $\tilde{\mathcal{P}}$ for $\tilde{K}B = \text{domain } \delta$.

A nonempty subcategory Φ^* of Φ is said to have the *domain enlargement property* (DEP) if for each $\phi \in \Phi^*$ and each $KP \to X$ which is an inclusion in $\tilde{\mathcal{P}}$ there exists a morphism $\phi \to \phi'$ in Φ^* with $U\nu : UP \subsetneq UD\phi'$. Dually the *range enlargement property* (REP) holds for Φ^* in Φ if for each $\phi \in \Phi^*$ and each inclusion $K\bar{P} \to Y$ of $\tilde{\mathcal{P}}$ there exists $\phi \to \phi''$ in Φ^* with $U\mu : U\bar{P} \subsetneq UR\phi''$.

We consider again the example in which $K^* : \mathcal{P}^* \to \mathcal{T}^*$ is the inclusion of the full subcategory $\mathcal{P}^*$ of finite groups belonging to the category of all reduced torsion abelian p-groups. Let $|\Phi|$ be the family of all height preserving isomorphisms from finite subgroups of X to finite subgroups of Y, where X and Y have the same Ulm invariants. It is a classical result of abelian group theory that given $\varphi \in |\Phi|$ and any element $a \in X$ and $b \in Y$ there is an isomorphism $\varphi' \in |\Phi|$ extending φ with $a \in \text{domain}\,\varphi'$ and $b \in \text{range}\,\varphi'$. In terms of the preceding paragraph and the category Φ_I of the first section this means that the domain and range enlargement properties hold for Φ_I in Φ.

A functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *final* if for each object B of $\mathcal{B}$ the category $(B \downarrow F)$ is nonempty and connected.

Theorem 2.3. *The functor $T_X : \Phi_D \to (\tilde{K} \downarrow X)$ is final provided that Φ_D has the domain enlargement property.*

Proof. We first show $(i \downarrow T_X)$ is nonempty for each $i \in |\tilde{K} \downarrow X|$. It follows from the hypotheses of the first section that i has a factorization as an isomorphism followed by an inclusion as in

$$\begin{tikzcd} \tilde{K}A \arrow[r] \arrow[dr, "\tilde{K}e"'] & X \\ & \tilde{K}B \arrow[u, "m"'] \end{tikzcd} \tag{2.4}$$

Furthermore there is an inclusion $k : B \to D\phi$ for some $\phi : D\phi \to R\phi$ in Φ_D since Φ_D is nonempty with DEP. The diagram

$$\begin{tikzcd} \tilde{K}B \arrow[r, "m"] \arrow[dr, "\tilde{K}k"'] & X \\ & \tilde{K}D\phi \arrow[u, "T_X\phi"'] \end{tikzcd} \tag{2.5}$$

commutes since the morphisms are inclusions and U is faithful. Thus $k \circ e : i \to T_X\phi$ in $(\tilde{K} \downarrow X)$.

Secondly, we must show that $(i \downarrow T_X)$ is connected for each i. Given $S_1, S_2 \in |i \downarrow T_X|$ it is clearly sufficient to show that there exists an object S and morphisms $U_i : S_i \to S$, $i = 1,2$ in $(i \downarrow T_X)$. In other words we wish to find an object $S : i \to T_X\phi''$ of $(i \downarrow T_X)$ and Φ_D morphisms $U_1 : \phi \to \phi''$ and $U_2 : \phi' \to \phi''$ such that

$$\begin{tikzcd} & T_X\phi \arrow[dr, "T_XU_1"] & \\ i \arrow[ur, "S_1"] \arrow[rr, dashed, "S"] \arrow[dr, "S_2"'] & & T_X\phi'' \\ & T_X\phi \arrow[ur, "T_XU_2"'] & \end{tikzcd} \tag{2.6}$$

commutes in $(\tilde{K} \downarrow X)$. Thus we are given $\tilde{\mathscr{P}}$ morphisms S_1 and S_2 such that the left triangles of

$$\begin{tikzcd} & \tilde{K}D\phi \arrow[d, "T_X\phi"'] \arrow[dr, "\tilde{K}T_XU_1"] & \\ \tilde{K}A \arrow[ur, "\tilde{K}S_1"] \arrow[r, "i"] \arrow[dr, "\tilde{K}S_2"'] & X & \tilde{K}D\phi \arrow[l, "T_X\phi''"'] \\ & \tilde{K}D\phi' \arrow[u, "T_X\phi'"] \arrow[ur, "\tilde{K}T_XU_2"'] & \end{tikzcd} \tag{2.7}$$

commute in $\mathscr{T}$. It is sufficient to find Φ_D morphisms $U_1 : \phi \to \phi''$ and $U_2 : \phi' \to \phi''$ since then the right triangles of (2.7) commute and hence so does (2.6) since $\tilde{K}$ is faithful and $T_X\phi''$ is monic. But by the domain enlargement property there is a morphism $U_1 : \phi \to \phi''$ in Φ_D for some ϕ'' with $U\gamma : UD\phi' \subseteq UD\phi''$. In the diagram

$$\begin{tikzcd} D\phi' \arrow[r, "\phi'"] \arrow[d, "\gamma"'] & R\phi' \arrow[d, "\delta"] \\ D\phi'' \arrow[r, "\phi''"] & R\phi'' \end{tikzcd} \tag{2.8}$$

let δ be the monic defined by $\delta = \phi''\gamma(\phi')^{-1}$. Then $U_2 = (\gamma, \delta)$ is in Φ_D as required.

Clearly Theorem 2.3 can be dualized. We simply let $T_Y : \Phi_R \to (\bar{K} \downarrow Y)$ be defined by $T_Y(\phi_i : D\phi_i \to R\phi_i) = (i_Y : KR\phi_i \to Y)$ and $T_Y(\alpha, \beta) = \beta$ and obtain

Theorem 2.9. *If Φ_R has the range enlargement property, then $T_Y : \Phi_R \to (\bar{K} \downarrow Y)$ is final.*

Theorem 2.10. *The inclusion functor $I_D : \Phi'_D \to \Phi_D$ is final provided that Φ'_D has both the domain and range enlargement properties in Φ.*

Proof. Since $(\phi \downarrow I_D)$ is clearly nonempty it is sufficient to show that any two objects S_1 and S_2 of $(\phi \downarrow I_D)$ can be connected by a sequence $S_1 \to S \leftarrow S_2$ of morphisms. That is, we must find morphisms $U_1 : \phi_1 \to \phi_3$ and $U_2 : \phi_2 \to \phi_3$ in Φ'_D such that

$$\begin{tikzcd} & I_D\phi_1 \arrow[dr, "I_D U_1"] & \\ \phi \arrow[ur, "S_1"] \arrow[rr, dashed, "S"] \arrow[dr, "S_2"'] & & I_D\phi_3 \\ & I_D\phi_2 \arrow[ur, "I_D U_2"'] & \end{tikzcd} \tag{2.11}$$

commutes in Φ_D. But by the range enlargement property in Φ'_D there exists $U_R : \phi_1 \to \bar{\phi}_1$ in Φ'_D with $R\phi_2 \subseteq R\bar{\phi}_1$. By the domain enlargement property in Φ'_D there exists $U_D : \bar{\phi}_1 \to \phi_3$ in Φ'_D with $D\phi_2 \subseteq D\phi_3$. Thus $U_1 = U_D U_R : \phi_1 \to \phi_3$ in Φ'_D and for $k : D\phi_2 \to D\phi_3$ the inclusion it is clear that the Φ_D morphism $(k, \phi_3 k \phi_2^{-1}) : \phi_2 \to \phi_3$ is in Φ'_D since $R\phi_2 \subseteq R\bar{\phi}_1 \subseteq R\phi_3$ where the latter inclusion holds since U_D is in Φ'_D. It is now clear that (2.11) commutes since Φ'_D is a preorder by Proposition 1.5.

Let $K : \mathcal{P} \to \mathcal{T}$ be the unique functor such that

$$\begin{tikzcd} \bar{\mathcal{P}} \arrow[r, "\bar{K}"] \arrow[d, "Q"'] & \bar{\mathcal{T}} \arrow[d, "Q"] \\ \mathcal{P} \arrow[r, "K"'] & \mathcal{T} \end{tikzcd} \tag{2.12}$$

commutes where, as in section one, $\mathcal{P}$ and $\mathcal{T}$ are preorders with canonical functors Q.

Theorem 2.13. *The unique functor $Q^* : (\bar{K} \downarrow X) \to (K \downarrow X)$ is final if Φ_D has the domain enlargement property. Furthermore*

$$\begin{tikzcd} (\bar{K} \downarrow X) \arrow[r, "Q^*"] \arrow[d, "D_X"'] & (K \downarrow X) \arrow[dl, "P_X"] \\ \mathcal{P} & \end{tikzcd} \tag{2.14}$$

commutes for P_X and D_X the canonical projection functors.

Proof. In order to show that $(\phi \downarrow Q^*)$ is nonempty and connected for all objects ϕ of $(K \downarrow X)$ it turns out that the crucial step is to show that given objects $S_1 : \phi \to Q^*\phi_1$ and $S_2 : \phi \to Q^*\phi_2$ of $(\phi \downarrow Q^*)$ there exists an object ϕ_3 and a pair $U_1 : \phi_1 \to \phi_3$ and $U_2 : \phi_2 \to \phi_3$ of morphisms in $(\bar{K} \downarrow X)$. In the diagram

$$\begin{tikzcd}
\tilde{K}B \arrow[r, "\tilde{K}\tilde{\phi}_1"] \arrow[dr, "\phi_1"'] & \tilde{K}\tilde{B} \arrow[d] \arrow[dr, "\tilde{K}i_B"] & \\
& X & \tilde{K}D\phi' \arrow[l, "\phi_3 = T_X\phi'"'] \\
\tilde{K}C \arrow[r, "\tilde{K}\tilde{\phi}_2"'] \arrow[ur, "\phi_2"] & \tilde{K}\tilde{C} \arrow[u] \arrow[ur, "\tilde{K}i_C"'] &
\end{tikzcd} \tag{2.15}$$

the left triangles represent the factorizations of ϕ_1 and ϕ_2 as isomorphisms followed by inclusions and $\phi' : D\phi' \to R\phi'$ is an object of Φ_D which, by the domain enlargement property, can be chosen so that $\tilde{B}$ and $\tilde{C}$ are contained in $D\phi'$. The triangles on the right commute since the morphisms are all inclusions. Using (2.15) we let $U_1 = i_B\tilde{\phi}_1$, $U_2 = i_C\tilde{\phi}_2$ and $\phi_3 = T_X\phi'$.

We remark that the same result holds for Q^* if we replace DEP for Φ_D by the requirement that any two subobjects of X of the type $\subseteq : KP \to X$ have an upper bound of the same type.

3. Shape isomorphisms and the back and forth property

The Pro-$\mathcal{P}$ isomorphism of the first section and the final functors of the second section can be combined to yield a shape isomorphism, if we start with an appropriate family Φ of isomorphisms. In order to see this we need the following general facts connecting the shape category of $K : \mathcal{P} \to \mathcal{T}$ and the Pro-$\mathcal{P}$ category. See [6] for details of the dual case.

There is a commutative diagram

$$\begin{tikzcd}
(\mathrm{Cat} \downarrow \mathcal{P}) \arrow[r, "\bar{M}"] & \text{Pro-}\mathcal{P} \\
(\mathrm{Cat} \downarrow \mathcal{P})_K \arrow[u, "I"] \arrow[r, "M"'] & \text{Pro}_K\text{-}\mathcal{P} \arrow[u, "\bar{I}"']
\end{tikzcd} \tag{3.1}$$

where the objects of $(\mathrm{Cat} \downarrow \mathcal{P})$ are functors with codomain $\mathcal{P}$ and morphisms $\theta : F \to G$ are functors from domain F to domain G such that $F = G\theta$. The categories $(\mathrm{Cat} \downarrow \mathcal{P})_K$ and $\text{Pro}_K\text{-}\mathcal{P}$ are the full subcategories of $(\mathrm{Cat} \downarrow \mathcal{P})$ and Pro-$\mathcal{P}$, respectively, whose objects are the projection functors $D_X : (K \downarrow X) \to \mathcal{P}$ with $D_X(KA \to X) = A$ on objects. We recall that the morphisms $X \to Y$ of the shape category S_K correspond to morphisms $D_X \to D_Y$ as in [6]. Let I and $\bar{I}$ be the inclusions and M the restriction of $\bar{M}$ where $\bar{M}$ is the identity on objects and on morphisms $\theta : F \to G$ the Pro-$\mathcal{P}$ morphism $\bar{M}\theta = (\psi, (f_i))$ where $\psi : |\text{domain } F| \to |\text{domain } G|$ is the function defined by $\psi j = \theta j$ and $f_i = 1 : Fj \to G\psi j$. The following theorem is obtained by dualizing results of Deleanu–Hilton in [6].

Theorem 3.2. *The functor M of diagram* (3.1) *is an isomorphism of categories and* *the functor $\bar{M}$ takes those morphisms of* (Cat $\downarrow \mathscr{P}$) *which are final functors to isomorphisms in* Pro-$\mathscr{P}$.

A Yoneda argument can be used to detect that certain shape morphisms always correspond to $\mathscr{T}$ morphisms.

Proposition 3.3. *The canonical functor $S : \mathscr{T} \to \mathscr{S}_K$ induces a bijection $\mathscr{T}(KP, X) \to \mathscr{S}_K(KP, X) =$ n.t.$(\mathscr{T}(K-, KP), \mathscr{T}(K-, X))$ for all objects P in $\mathscr{P}$ and X in $\mathscr{T}$ provided that $K : \mathscr{P} \to \mathscr{T}$ is full (or, more generally, rich in the sense of* [7]).

The following theorem shows how Section 2 may be used in conjunction with (3.2) to obtain Pro-$\tilde{\mathscr{P}}$ isomorphisms connecting the comma categories $(\tilde{K} \downarrow X)$ and $(\tilde{K} \downarrow Y)$ with our class $|\Phi|$ of isomorphisms between $\tilde{\mathscr{P}}$ subobjects of X and Y.

Theorem 3.4. *There are* Pro-$\tilde{\mathscr{P}}$ *isomorphisms*

$$\begin{tikzcd} \Phi'_D \arrow[d, "\tilde{D}'"'] \arrow[r, phantom, "\overset{\tilde{\rho}_X}{\Rightarrow}"] & (\tilde{K} \downarrow X) \arrow[d, "\tilde{D}_X"] \\ \tilde{\mathscr{P}} & \tilde{\mathscr{P}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Phi'_R \arrow[d, "\tilde{R}'"'] \arrow[r, phantom, "\overset{\tilde{\rho}_Y}{\Rightarrow}"] & (\tilde{K} \downarrow Y) \arrow[d, "\tilde{D}_Y"] \\ \tilde{\mathscr{P}} & \tilde{\mathscr{P}} \end{tikzcd} \tag{3.5}$$

provided that the domain and range enlargement properties hold for Φ_I in Φ, or more generally, if they hold for Φ'_D in Φ and Φ'_R in Φ.

Proof. From (2.2), (2.3) and (2.10) it follows that there is a commutative diagram

$$\begin{tikzcd} \Phi'_D \arrow[r, "I_D"] \arrow[dr, "\tilde{D}'"'] & \Phi_D \arrow[r, "T_X"] \arrow[d, "\tilde{D}"] & (\tilde{K} \downarrow X) \arrow[dl, "\tilde{D}_X"] \\ & \tilde{\mathscr{P}} & \end{tikzcd} \tag{3.6}$$

with $T_X I_D$ final. Similarly $T_Y I_R : \Phi'_R \to (\tilde{K} \downarrow Y)$ is final. From (3.2) it follows that $\tilde{\rho}_X = \bar{M}(T_X I_D) : \tilde{D}' \to \tilde{D}_X$ and $\tilde{\rho}_Y = \bar{M}(T_Y I_R) : \tilde{R}' \to \tilde{D}_Y$ are Pro-$\tilde{\mathscr{P}}$ isomorphisms.

Corollary 3.7. *Let $K^* : \mathscr{P}^* \to \mathscr{T}^*$ have associated functors $\tilde{K} : \tilde{\mathscr{P}} \to \tilde{\mathscr{T}}$ and $K : \mathscr{P} \to \mathscr{T}$ and suppose that DEP and REP hold for Φ_I in Φ. Then*

(a) *X and Y are isomorphic in $\mathscr{S}_K$ if and only if there exists a* Pro-$\tilde{\mathscr{P}}$ *isomorphism from $\tilde{D}'$ to $\tilde{R}'$.*

(b) *X and Y are isomorphic in $\mathscr{S}_K$.*

Proof. If $\tilde{\rho} : \tilde{D}' \to \tilde{R}'$ is a Pro-$\tilde{\mathscr{P}}$ isomorphism, then by (3.4) $\tilde{\rho}_Y \circ \tilde{\rho} \circ (\tilde{\rho}_X)^{-1}$ is a Pro-$\tilde{\mathscr{P}}$ isomorphism in $\mathrm{Pro}_{\bar{K}}$-$\tilde{\mathscr{P}}$ which must equal $M\theta$ for some θ in $\mathscr{S}_K =$ (Cat $\downarrow \tilde{\mathscr{P}})_K$ by (3.2). Conversely, if $\theta : X \to Y$ is an $\mathscr{S}_K$ isomorphism then

$M\theta : \tilde{D}_X \to \tilde{D}_Y$ by (3.2) and $(\tilde{\rho}_Y)^{-1} \circ M\theta \circ \tilde{\rho}_X : \tilde{D}' \to \tilde{R}'$ is an isomorphism in Pro-$\tilde{\mathcal{P}}$ by (3.4). Thus (a) holds. The functor $\delta_X = Q^* T_X I_D : \Phi'_D \to \Phi_D \to (\tilde{K} \downarrow X) \to (K \downarrow X)$ is final by (2.3), (2.10) and (2.13). From (2.14) and (3.6) it is clear that $\delta_X : D' = Q\tilde{D}' \to P_X$ in (Cat $\downarrow \mathcal{P}$). Hence $\bar{M}(\delta_X)$ is a Pro-$\mathcal{P}$ isomorphism $D' \to P_X$ by (3.2) and there is a similarly obtained isomorphism $\bar{M}(\delta_Y) : R' \to P_Y$. Combining these with the Pro-$\mathcal{P}$ isomorphism $g : D' \to R'$ of (1.6) we obtain an isomorphism $P_X \to P_Y$ in Pro$_K$-$\mathcal{P}$ and hence by (3.2) an isomorphism $X \to Y$ in $\mathcal{S}_K \approx$ (Cat $\downarrow \mathcal{P})_K$.

Under the hypotheses of (3.7) (a) above we remark that a Pro-$\tilde{\mathcal{P}}$ isomorphism will sometimes exist, as shown by (1.10), and sometimes one cannot exist, as shown in (3.15).

We next characterize the isomorphisms of the category $\mathcal{S}_K$ in terms of the isomorphism classes $|\Phi|$ of $\mathcal{P}^*$ isomorphisms.

Theorem 3.8. *Let $K^* : \mathcal{P}^* \to \mathcal{T}^*$ have associated functor $K : \mathcal{P} \to \mathcal{T}$ and suppose that any two inclusions $K^*P \to X$ and $K^*P' \to X$ in $\mathcal{T}^*$ have an upper bound of the same type. Then X and Y are isomorphic in $\mathcal{S}_K$ if and only if there is a family $|\Phi|$ of $\mathcal{P}^*$ isomorphisms between subobjects of X and Y under inclusion such that* DEP *holds for Φ_D and* REP *holds for Φ_R.*

Proof. Suppose $\theta : X \to Y$ is an $\mathcal{S}_K$ isomorphism. Let $|\Phi|$ be the family of all $\mathcal{P}^*$ isomorphisms $\phi : D\phi \to R\phi$ for which inclusions $T_X\phi : KD\phi \to X$ and $T_Y\phi : KR\phi \to Y$ exist. Given $\phi \in |\Phi|$ and an inclusion $KA \to X$ we can by hypothesis factor $T_X\phi : KD\phi \to KA' \to X$ for some A' with $A \subseteq A'$. Then we regard $KA' \to X$ successively as a $\mathcal{T}$ and then an $\mathcal{S}_K$ morphism. Comparing with $\theta : X \to Y$ in $\mathcal{S}_K$ we obtain $KA' \to Y$ in $\mathcal{S}_K$ which can be regarded as a $\mathcal{T}$ morphism $\kappa : KA' \to Y$ by (3.3). Choosing a representative $\tilde{\kappa}$ of κ in $\tilde{\mathcal{T}} \subseteq \mathcal{T}^*$ we then obtain a diagram

$$\begin{tikzcd} & Y & \\ KA' \arrow[ur, "\tilde{\kappa}"] \arrow[rr, "K\delta"'] & & KB' \arrow[ul, "\tilde{\rho}"'] \\ KD\phi \arrow[u, "K\alpha"] \arrow[rr, "K\beta"'] & & KB \arrow[u, "K\gamma"'] \end{tikzcd} \tag{3.9}$$

commuting in $\tilde{\mathcal{T}}$ where $\tilde{\rho} \circ K\delta$ is the factorization of $\tilde{\kappa}$ as isomorphism followed by inclusion and $K\gamma \circ K\beta$ is a similar factorization of $K\delta \circ K\alpha$. Thus

$$\begin{tikzcd} A' \arrow[r, "\delta"] & B' \\ D\phi \arrow[u, "\alpha"] \arrow[r, "\phi"'] & R\phi \arrow[u, "\gamma \circ \beta \circ \phi^{-1}"'] \end{tikzcd} \tag{3.10}$$

commutes in $\mathfrak{F}$ with $(\alpha, \gamma \circ \beta \circ \phi^{-1}) : \phi \to \delta$ in Φ_D with $A \subseteq A'$. Thus Φ_D has DEP. To see that Φ_R has REP we simply do the same proof using the family of inverses of $|\Phi|$ members. Conversely, given such a family we define a functor $\theta : (K \downarrow X) \to (K \downarrow Y)$ by $\theta(KA \to X) =$ the unique map $KA \to Y$ in $\mathcal{T}$ and verify that DEP for Φ_D insures that such a functor exists. Then use REP for Φ_R to insure the existence of a functor in the other direction.

We now relate the preceding results to the back and forth lemma. Accordingly we let X and Y be algebraic structures of the same type τ, as in Mal'cev [10]. In brief, this means that X and Y are sets with the same number of operators and predicates of the same arities.

Let $K^* : \mathcal{P}^* \to \mathcal{T}^*$ be the inclusion of the category of finitely generated algebras of type τ in the category of all such algebras. Classically, a family $|\Phi|$ of $\mathcal{P}^*$ isomorphisms between subobjects of X and Y is said to have the *back and forth property* if for each $f \in |\Phi|$, $a \in X$ and $b \in Y$ there is $g \in |\Phi|$ with $f \subseteq g$, $a \in \text{domain}\, g$ and $b \in \text{range}\, g$.

Proposition 3.11. *Let $K^* : \mathcal{P}^* \to \mathcal{T}^*$ be the inclusion of the category of finitely generated algebras of type τ in that of all τ algebras. Then the back and forth property for a family $|\Phi|$ of $\mathcal{P}^*$ isomorphisms between subobjects of X and Y is equivalent to the domain and range enlargement properties for Φ_I.*

Corollary 3.12. *If a family $|\Phi|$ of $\mathcal{P}^*$ isomorphisms between subobjects of the τ-algebras X and Y has the back and forth property, then X and Y are isomorphic in the shape category $\mathcal{S}_K$ for $K : \mathcal{P} \to \mathcal{T}$ the functor associated with K^*.*

Proof. Follows from (3.7).

Corollary 3.13. *If the algebras X and Y are $\infty\omega$ equivalent in the sense of* [3], *then they are isomorphic in $\mathcal{S}_K$.*

Proof. This is immediate from (3.12) plus the observation of Barwise [3] that X and Y are $\infty\omega$ equivalent if and only if the back and forth property holds for suitable $|\Phi|$.

We now close with a few illustrative examples.

Example 3.14. Let $K^* : \mathcal{P}^* \to \mathcal{T}^*$ be the inclusion of the category of finite sets in that of all sets. Let $|\Phi|$ be the set of all isomorphisms between finite subsets of two given sets X and Y. If X and Y are infinite, then clearly $|\Phi|$ has the back and forth property and X and Y are isomorphic in $\mathcal{S}_K$ by (3.12).

Proposition 3.15. *If X and Y are infinite sets of different cardinality, then they are isomorphic in $\mathcal{S}_K$ but not in $\mathcal{S}_{K^*}$* (or $\mathcal{S}_{\bar{K}}$) *for K the functor associated with the functor K^* of* (3.12).

Proof. Suppose

$$\theta : ((K^* \downarrow X) \xrightarrow{D_X} \mathscr{P}^*) \longrightarrow ((K^* \downarrow Y) \xrightarrow{D_X} \mathscr{P}^*)$$

is an isomorphism in $\mathscr{S}_{K^*}$. Given $x \in X$ the inclusion $\alpha : \{x\} \to X$ is an object of $(K^* \downarrow X)$ and $\theta\alpha : \{x\} \to Y$ is an object of $(K^* \downarrow Y)$. Thus θ determines a function $X \to Y$ with inverse determined by θ^{-1}.

Thus it is immediate from (3.7) (a) and (3.11) that the existence of a Pro-$\mathscr{P}$ isomorphism $g : Q\bar{D} \to Q\bar{R}$ shown in (1.6) cannot in general be replaced by the stronger result that there exists a Pro-$\bar{\mathscr{P}}$ isomorphism $\bar{g} : \bar{D} \to \bar{R}$.

Example 3.16. Let $\mathscr{P}^*$ be the category of all sets of cardinality less than a given infinite cardinal d and let $K^* : \mathscr{P}^* \to \mathscr{T}^*$ be the inclusion functor to the category of sets. If $|\Phi|$ is the set of all isomorphisms between subsets of cardinality less than d in X and Y, then Φ_I has the domain and range enlargement properties and X and Y are isomorphic in $\mathscr{S}_K$ by (3.7) provided that X and Y are infinite of cardinalities $\geq d$. Of course, by (3.3), shape isomorphism is the same as ordinary isomorphism for cardinality $< d$.

Example 3.17. Let $\mathscr{T}^*$ be the category of superatomic Boolean algebras and $\mathscr{P}^*$ the full subcategory of finite algebras. Suppose that $|\Phi|$ is the family of all isomorphisms between finite subalgebras of X and Y in $\mathscr{T}^*$. If X and Y are infinite and have the same Day invariants, then $|\Phi|$ has the back and forth property (see [1]) and hence X and Y are isomorphic in $\mathscr{S}_K$.

Example 3.18. Let $K^* : \mathscr{P}^* \to \mathscr{T}^*$ be the inclusion of the full subcategory of finite groups belonging to the category $\mathscr{T}^*$ of reduced torsion abelian p groups for a fixed prime p. If X and Y are infinite and have the same Ulm invariants, then the family $|\Phi|$ of all height preserving isomorphisms between finite subgroups of X and Y has the back and forth property (see [3] and [4]) and X and Y are isomorphic in $\mathscr{S}_K$ by (3.12).

We remark that (3.3) shows us how to reconstruct $\mathscr{T}$ morphisms (and hence $\mathscr{T}^*$ morphisms) from $\mathscr{S}_K$ morphisms when the domain is in $|\mathscr{P}^*| = |\mathscr{P}|$.

We end with a result going back to Cantor (cf. [3]).

Example 3.19. Let $K^* : \mathscr{P}^* \to \mathscr{T}^*$ be the inclusion of the full subcategory of finite linearly ordered sets in all such sets. If X and Y are dense linearly ordered sets and $|\Phi|$ is the set of all isomorphisms of a finite subordering of X onto a finite subordering of Y, then $|\Phi|$ has the back and forth property and $X \cong Y$ in $\mathscr{S}_K$.

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